On the Application of Homeomorphism on Amoeba Proteus*

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Abstract

Even though mathematics and biology are seen as different areas of study, it does not dispute the fact that some concepts can be applied on them from one to the other. The term homeomorphism, from the topological field of study, was thoroughly explicated using amoeboid movement. The topologies ($\tau_i\ for\ i = 1, 2$) are three-point sets $X = \{a, b, c\}$ of amoeba at certain positions on the surface of the X. The initial state of amoeba is the topological space (X, τ_1) and the transformed state is (Y, τ_2). The amoeba in a certain topological space (X, τ_1) is transformed to another topological space (Y, τ_2), with bi-continuous function $f: X \rightarrow Y$ and $f^{-1}: Y \rightarrow X$ such that $\tau_1 \rightarrow \tau_2\ \forall\ x\epsilon X, \exists\ y\epsilon Y$. The topological spaces (X, τ_1) and (Y, τ_2) are homeomorphic or topologically equivalent since there is continuous invertible function $f : X \rightarrow Y$ with continuous inverse $f^{-1} : Y \rightarrow X$.

Keywords: Homeomorphism, Topological Space, Amoeboid Movement, Bi-Continuous Function

1 Introduction

Topology is the study of shapes, including their properties, deformations applied to them, mappings between them and configurations composed from them (Carlson, 2017). Most of the researches performed on homeomorphism are based on the mathematical aspects with the idea of topology. Examples of homeomorphism are mostly conducted in the abstract world where it normally emphasizes on mathematical terms. It becomes very difficult for a layman to understand a work or project with the concept of homeomorphism. The mathematical equations, symbols, terms and other visuals seems cumbersome and a headache to others no matter how short or easy it is for a topologist (Anon., 2019). But it was established that a homeomorphism between two topological spaces X and Y is a bijective map (Caldas *et al.,* 2009). And generically, there is but One-Self Homeomorphism of the Contour Set (Aikin, 2006). Other studies succeeded in explaining related concepts such as metric space, hamming distance, Levenshtein distance, with real life applications. DNA hybridity was also used to demonstrated the applicability of metric space (Zigli *et al.,*2020). But all these works seem cumbersome and very difficult to understand. This study seeks to explain homeomorphism in simple and clears terms. In this paper we explained and demonstrated the application of homeomorphism using a unicellular organism, Amoeba.

2 Resources and Methods Used

2.1 Function

2.1.1 Definition

Let X and Y be sets. A function f from X to Y is a relation between X and Y that associates to each x in X a unique y in Y. We write $\boldsymbol{f(x) = y}$ and call y the image of x under f. The set X is called the domain of the function and the set Y is called the range of the function.

The notation $f: X \rightarrow Y$ is used to indicate that f is a function from X to Y. We also call a function $f: X \rightarrow Y$ a mapping and say that f maps X to Y. Further, if $f(x) = y$, then we say that f maps x to y.
Definition 2.1.2: For a function $f: X \rightarrow Y$ and a subset A of X, define the image of A under f to be the set $f\ (A) = \{y\epsilon Y \mid y = f\ (x)$ for $x\epsilon A\}$, as illustrated in Fig. 1. The set $f(X)$ is the image of the domain; we also refer to it as the image of f.

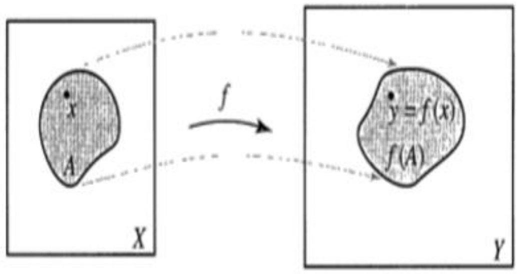

Fig. 1 The Image of A under *f*

Theorem 2.1.3: If f: $\boldsymbol{X} \rightarrow \boldsymbol{Y}$ is a function and A and B are subsets of X, then;

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(i) $f(A \cup B) = f(A) \cup f(B)$
(ii) $f(A \cap B) = f(A) \cap f(B)$
(iii) $f(A) - f(B) = f(A - B)$

Theorem 2.1.4: If f: $X \to Y$ is a function, V and W are subsets of Y, then,

(i) $f^{-1}(V \cup W) = f^{-1}(V) \cup f^{-1}(W)$
(ii) $f^{-1}(V \cap W) = f^{-1}(V) \cap f^{-1}W)$
(iii) $f^{-1}(V - W) = f^{-1}(V) - f^{-1}(W)$

2.2 Continuous Functions

2.2.1 Definition

A function $f: X \to Y$ is continuous if the pre-image of every open set in Y is open in X.

However, checking that every open set in Y has an open pre-image in X is more than we really need to do. As theorem 2.2.2 indicates below, to prove that f is continuous, it suffices to consider only the sets in a fixed basis for Y, showing that the pre-image of each basis element is open in X.

2.2.2 Theorem

Let X and Y be topological spaces and B be a basis for the topology on Y. Then $f: X \to Y$ is continuous if and only is open in X for $B \epsilon \beta$.
Proof. Suppose $f: X \to Y$ is continuous. Then $f^{-1}(V)$ is open in X for every V open in Y. Since every basis element B is open in Y, it follows that f^{-1} (B) is open in X for all $B \epsilon \beta$. Now, suppose $f^{-1}(B)$ is open in X for $B \epsilon \beta$. We show that f is continuous. Let V be an open set in Y. Then V is a union of basic elements, say $V = \cup (Ba)$ thus,
$$f^{-1}(V) = f^{-1}(\cup Ba) = \cup f^{-1}(Ba)$$
By assumption, each set (Ba) is open in X; therefore, so is their union. Thus, (V) is open in X, and it follows that the pre-image of every open set in Y is open in X. Hence, f is continuous.
Example 2.2.3
The functions $f(y) = y + 5$ and $g(y) = 3y$ are all continuous functions mapping $\mathbb{R}$ to $\mathbb{R}$ in the standard topology. Let (a, b) with $a < b$ be a basis element for a standard topology. Then,
$$f^{-1}((a, b)) = (a - 5, b - 5)$$
$$f^{-1}(a, b) = (a/3, b/3)$$ (Colin, 2008).

2.3 Homeomorphism

2.3.1 Definition

A function f between two topological spaces is a homeomorphism if it satisfies the following properties:

a) f is a bijection (one-to-one and onto),
b) f is continuous,
c) the inverse function is continuous (is an open mapping).

A continuous bijective function that has a continuous inverse is called a homeomorphism. Such functions provide us with the main notion of topological equivalence (Acheson *et al.*, 2017).

2.4 Amoeba Proteus

Amoeba proteus is a eukaryotic unicellular species known as Protista in the World. Amoebas are amorphous and look like jelly-like blobs. Such microscopic protozoa move by changing their form, showing a peculiar type of crawling motion which has become known as amoeboid motion (Aparna, 2016).

Simply put, amoeba is a unicellular organism which has the ability to alter its shape, primarily by extending and retracting its pseudopods.

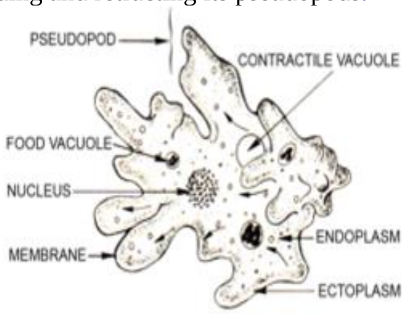

Fig. 2 Morphology of Amoeba

3 Results and Discussion

3.1 Illustration 1

Suppose X is an amoeba in a topological space (X, τ_1) where X is a non-empty set with the element $X = \{N_1, M_1, V_1\}$. $N_1, M_1,$ and V_1 are certain parts of the amoeba X at a specific location in the eukaryotic cell. So, we defined topology on set $X = \{N_1, M_1, V_1\}$ as: $\tau_1 = \{X, \emptyset, \{N_1\}, \{M_1\}, \{V_1\}, \{N_1, V_1\}, \{M_1, V_1\}, \{N_1, M_1\}\}$ where τ_1 represents topology on X and τ_1 satisfies all the conditions/ properties of topology. But in this work, our aim is to apply homeomorphism on the amoeboid movement. From definition 2.3.1, we understood that homeomorphism refers to continuous bijection from one topological space to another with continuous inverse. The figure below illustrates the idea of homeomorphism where amoeba X was transformed to amoeba Y.

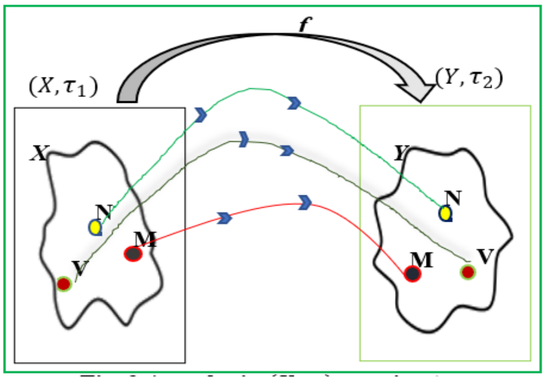

Fig. 3 Amoeba in (X, τ_1) moving to (Y, τ_2)

From Fig. 3, we could see amoeba X which has been transformed through continuous deformations from the topological space (X, τ_1) to another topological space (Y, τ_2). Y is the new shape of the amoeba by twisting, bending and shrinking amoeba X. The bending, twisting and shrinking are noted for the propagation of amoeba X from the initial position to the final position. It was assumed that the amoeba transformed to the final location with the function $y = f(x) = \frac{x}{1+|x|}$ where $f : X \to Y$ and $f : Y \to X \ \forall \ x \epsilon X, \exists \ y \epsilon Y : \tau_1 \longrightarrow \tau_2$. The inverse of the function $f(x)^{-1} = f(y) = \frac{x}{1-|x|}$. Thus, the inverse function is also continuous.

The elements of set $Y = \{N_1, M_1, V_1\}$, where
N_2 = Nucleus at final location
M_2 = Mitochondria at final location
V_2 = Vacuole at final location.

We then defined topology on set Y as $\tau_2 = \{Y, \emptyset, \{N_2\}, \{M_2\}, \{V_2\}, \{N_2, V_2\}, \{M_2, V_2\}, \{N_2, M_2\}\}$
Set X was mapped into set Y where each $x \in X$ mapped onto a unique $y \in Y$.

$$\begin{array}{l} \tau_1 = \{\boldsymbol{X}, \emptyset, \{\boldsymbol{N_1}\}, \{\boldsymbol{M_1}\}, \{\boldsymbol{V_1}\}, \{\boldsymbol{N_1}, \boldsymbol{V_1}\}, \{\boldsymbol{M_1}, \boldsymbol{V_1}\}, \{\boldsymbol{N_1}, \boldsymbol{V_1}\}\} \\ \quad \Downarrow \ \Downarrow \ \Downarrow \ \Downarrow \ \Downarrow \ \Downarrow \ \Downarrow \ \Downarrow \\ \tau_2 = \{\boldsymbol{Y}, \emptyset, \{\boldsymbol{N_2}\}, \{\boldsymbol{M_2}\}, \{\boldsymbol{V_2}\}, \{\boldsymbol{N_2}, \boldsymbol{V_2}\}, \{\boldsymbol{M_2}, \boldsymbol{V_2}\}, \{\boldsymbol{N_2}, \boldsymbol{M_2}\}\} \end{array}$$

So, $f(N_1) = N_2$, $f(M_1) = M_2$ and $f(V_1) = V_2$
Since all the mappings between the two topological properties were preserved, it implies X and Y are topologically equivalent. Thus X is homeomorphic to Y.

3.2 Illustration 2

In this illustration, it was assumed that the amoeba is a geometric object such as a sphere which was trasnformed into a cube (Idris *et al*, 2008). Suppose the surface of the sphere is in two manifolds or two dimensions with a topology defined on three particles or elements of the spherical amoeba, then the transformation can be done using homeomorphism as follows:

Let S be the spherical amoeba. Set $S = \{X_1, Y_1, Z_1\}$,where X_1 = Nucleus at initial location, Y_1 = Mitochondria at initial location and Z_1 = Contractile vacuole at initial location.

Note: The surface of a sphere contains infinitely many points but we are just considering only three points where three elements were sited.

Let $\tau_1 = \{S, \emptyset, \{X_1\}, \{Y_1\}, \{Z_1\}, \{X_1, Y_1\}, \{Y_1, Z_1\}, \{X_1, Z_1\}\}$, where τ_1 is the topology on S and τ_1 satisfies all the conditions/properties of topology. The spherical amoeba moved from the initial topological space (S, τ_1) to another topological space (C, τ_2) where S was transformed to F (cubic amoeba). The amoeba moved with bi-continuous function $c = f(s) = \frac{s^2-16}{s-4}$, where $f : S \to C$ and $f^{-1} : C \to S \ \forall \ s \in S, \exists \ y \epsilon Y : \tau_1 \longrightarrow \tau_2$. The inverse of the function is $s = f(c) = c - 4$. Thus, f is bi-continuous.

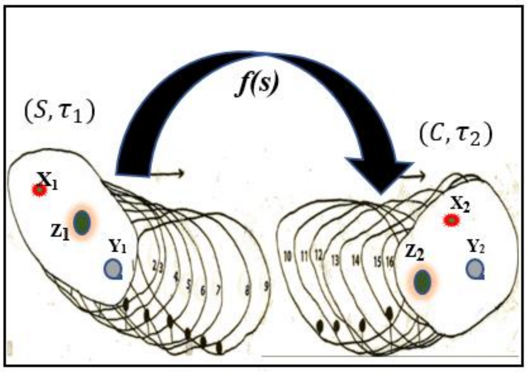

Fig. 4 A Sherical Amoeba Moving from One Topological Space to Another

From Fig. 4, elements of $C = \{X_2, Y_2, Z_2\}$, where, X_2 = Nucleus at final location, Y_2 = Mitochondria at final location and Z_2 = Vacuole at final location.
The topology on C is defined as $\tau_2 = \{C, \emptyset, \{X_2\}, \{Y_2\}, \{Z_2\}, \{X_2, Z_2\}, \{Y_2, Z_2\}, \{X_2, Y_2\}\}$.
Each element in X is mapped on to a unique element in Y such that $f(X_1) = X_2$, $(Y_1) = Y_2$ and $f(Z_1) = Z_2$. Thus

$$\begin{array}{l} \tau_1 = \{\boldsymbol{S}, \emptyset, \{\boldsymbol{X_1}\}, \{\boldsymbol{Y_1}\}, \{\boldsymbol{Z_1}\}, \{\boldsymbol{X_1}, \boldsymbol{Z_1}\}, \{\boldsymbol{Y_1}, \boldsymbol{Z_1}\}, \{\boldsymbol{X_1}, \boldsymbol{Y_1}\}\} \\ \quad \Downarrow \ \Downarrow \ \Downarrow \ \Downarrow \ \Downarrow \ \Downarrow \ \Downarrow \ \Downarrow \\ \tau_2 = \{\boldsymbol{C}, \emptyset, \{\boldsymbol{X_2}\}, \{\boldsymbol{Y_2}\}, \{\boldsymbol{Z_2}\}, \{\boldsymbol{X_2}, \boldsymbol{Z_2}\}, \{\boldsymbol{Y_2}, \boldsymbol{Z_2}\}, \{\boldsymbol{X_2}, \boldsymbol{Y_2}\}\} \end{array}$$

In Fig. 4, (S, τ_1) is the topological space of the amoeba when it was spherical. And (C, τ_2) is the topological space of the transformed spherical shape into a cube. The figures 1,2,3,16 represent the series of transformations that amoeba S went through before finally transforming into a cube, C. The function $y = f(x) = \frac{x}{1+|x|}$ where $f : X \to Y$ and $f : Y \to X \ \forall \ x \epsilon X, \exists \ y \epsilon Y : \tau_1 \longrightarrow \tau_2$ was used since f and f^{-1} are continuous and bijective.

Though amoeba does not have a regular shape, it can assume any shape. The different shapes are topologically identical, since there is a continuous deformation from one to another.

4 Conclusions

Homeomorphism was successfully appplied to the amoeboid movement and thoroghly explained. It was seen that, the homeomorphism between the two topological spaces preserved all the topological properties defined on the amoeba. The diagrams and basic concepts used make it more comprehensible and self-explanatory for the science world.

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