Comparative Study of Mathematical Models for Population Growth in Ghana Prisons*

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Abstract

The study examined the population growth in Ghana Prisons based on four growth models. The models examined were the Logistic, Gompertz, Gaussian and the Richards. All the models were then used to predict the population growth in the Ghana prisons up to 2025. Although all the models gave a good estimation of the observed population growth of prisoners in Ghana, the Gompertz model was adjudged to give the best approximation. The model was selected based on its high proportion of variance as well as having the least value in terms of Root Mean Square Error (RMSE) and information loss. The model suggested that the population of prisoners in Ghana could increase to about 14 530 by 2025.

Keywords: Sigmoid, Logistic, Gompertz, Gaussian, Richards, Prisoners, Population Growth

1 Introduction

Over the years, the population growth of Ghana Prisons has been on the ascendancy. According to a research conducted in 2018 by World Prison Brief (WPB) and Institute of Criminal Policy Research (ICPR), the population growth of Ghana prisons from 1982 to 2017 has been on the rise with some fluctuations in growth from 2009 to 2011. For instance, in 1982, the population of Ghana prisons was 4 852 inmates. This number has increased tremendously over the years. As at 1995 the population has increased to 7 727 inmates. In 2000 the population increased to 9 507 inmates, and similarly in 2017 it increased to 14 417 inmates (Anon. 2018; Nutsukpui, 2011).

Also, from a statistical report provided by Ghana Prisons Service, Ghana's prisons are overcrowded by 45.5% (Anon., 2016). Most of the prison structures, which are already in a deplorable state, are housing more inmates than the number designed for the structures. Though these penitentiaries are expected to be reformatory homes for prisoners, these prisoners come out hardened than they were due to the deplorable conditions. Various corporate bodies have agitated the need for both the state and private organisations to come and help turn the situation around. Due to this, the government of Ghana has decided to embark on a program dubbed Project Efiase (Anon., 2015). This program is aimed at expanding Ghanaian prisons and improving the living conditions in the prisons to accommodate increasing population of prisoners.

For such a program to be successful, it is imperative for policy makers to have a fair knowledge of the population growth of prisoners and be able to predict into the future the expected number of inmates and the facilities required to host them.

In the field of mathematical approximations, sigmoid functions offer a wide range of applications such as modelling growth processes of biological, economic phenomena (Hernandez-Llamas and Ratkowsky, 2004; Müller and Dirner, 2010; Appiah *et al*., 2016).

In this paper, sigmoid functions are used to approximate the cumulative population growth in Ghana prisons which would be used to predict the number of prisoners in Ghana's prisons in the near future. The functions examined are the Logistic, Gompertz, Gaussian and Richards. These functions have frequently been used in literature due to their good approximation capabilities to describe growth, because of their nonlinear structures (Unal *et al*., 2018).

2 Resources and Methods Used

To approximate the cumulative population growth of prisoners in Ghana, annual data of the population of prisoners was obtained from Ghana Prisons Service. The data span from 2000 to 2017. Fig. 1 shows the time series plot of the population of prisoners in Ghana. As observed, the growth rate of prisoners increased rapidly from 2000 until 2009 where the growth rate started to decline to 13 396 in 2011. Fig. 1 also shows that the highest growth of prisoners occurred in 2014 with a value of 14 599. The number decreased in 2015 but increased again in 2016 and 2017.

The cumulative population growth of prisoners in Ghana, shown in Fig. 1 was approximated by four different growth models: the Logistic, Gompertz,

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Gaussian and the Richards model. The mathematical approximations were estimated using the non-linear module of the JMP statistical software, version 10.0. The models were fitted using the least squares loss function. The procedure minimises the sum of the loss function across the specified given observations. The fitted models and estimated parameters are then compared to determine the best approximated model.

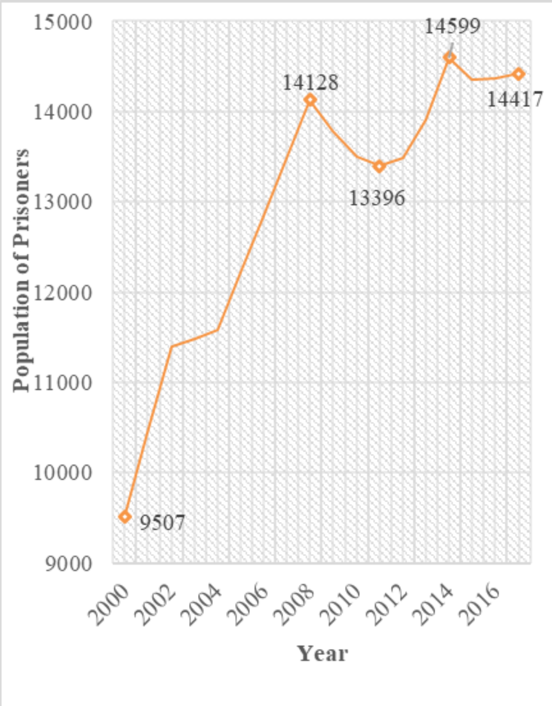

Fig. 1 Observed Annual Population of Prisoners

2.1 Logistic Model

The Logistic growth model (referred to as the Verhulst model) was first proposed as a model for population growth by Verhulst (1838). The model has the form as shown in Equation (1).

$$P(t_i) = \frac{K}{1+\mathrm{Exp}\left[-A(t_i - r)\right]} \tag{1}$$

where $P(t_i)$ is the population of prisoners at time t_i, K is the maximum cumulative population of prisoners; A and r are the growth rate and inflection point respectively. The unknown parameters K, A and r are approximated to fit the observed data points shown in Fig. 1.

2.2 Gompertz Model

The Gompertz model was originally derived to estimate human mortality by Gompertz (1825). The model is shown in Equation (2).

$$P(t_i) = K * \mathrm{Exp}\left[-\mathrm{Exp}\left(-A(t_i - r)\right)\right] \tag{2}$$

where $P(t_i)$ is the population of prisoners at time t_i, K is the maximum cumulative population of prisoners (carrying capacity), A and r are the growth rate and inflection point respectively. The unknown parameters K, A and r are approximated to fit the observed data points shown in Fig. 1.

2.3 Gaussian Model

The Gaussian normal distribution is widely used in probability calculations (Hagen *et al.*, 2007; Guo, 2011). The density function is given by Equation (3)

$$P(t_i) = K * \mathrm{Exp}\left[-\left[0.5\left(\frac{(t_i - r)}{A}\right)^2\right]\right] \tag{3}$$

where $P(t_i)$ is the population of prisoners at time t_i, K is the peak value; A and r are the growth rate and inflection point respectively. The unknown parameters K, A and r are approximated to fit the observed data points shown in Fig. 1.

2.4 Richards Model

The Richards model is widely used in modelling and estimating growth parameters (Catlin *et al.*, 2013; Aghababaie *et al.*, 2014; Tjørve and Tjørve, 2017). The model has the form as shown in Equation (4).

$$P(t_i) = L + (K - L) * \mathrm{Exp}\left[-\mathrm{Exp}\left(-A(t_i - r)\right)\right] \tag{4}$$

where $P(t_i)$ is the population of prisoners at time t_i, L and K are the lower and upper asymptote; A and r are the growth rate and inflection point respectively. The unknown parameters K, A and r are approximated to fit the observed data points shown in Fig. 1.

2.5 Validation and Diagnostic Checking of Models

2.5.1 Coefficient of Determination (R^2)

R^2 is a widely used measure of the goodness of fit of the regression relationship, showing how well a model represents the observed data. R^2 is given as Equation (5).

$$R^2 = \frac{SSR}{SST} \tag{5}$$

where SSR and SST are the regression sum of squares and total sum of squares respectively.

2.5.2 Akaike Information Criteria Corrected (AICc)

AIC estimates the expected information loss incurred when a probability distribution associated with the true data-generating process is approximated by a probability distribution associated with the model under evaluation. Hence, the model with the lowest AIC is the model with the smallest expected information loss (Akaike, 1973). In practice, particularly when sample sizes are small AICc, a second order bias correction for AIC is appropriate to use (Burnham *et al*., 2011). AICc is estimated using Equation (7)

$$AIC = -2\ln p(y \mid \hat{\theta}) + 2k \tag{6}$$

$$AICc = AIC + \frac{2k^2 + 2k}{n-k-1} \tag{7}$$

where $\ln p(y \mid \hat{\theta})$ is the log maximum likelihood that quantifies goodness-of-fit, $2k$ is a penalty for model complexity, measured by the number of adjustable model parameters k and n is the sample size.

2.5.3 Root Mean Square Error (RMSE)

RMSE is among the most commonly used scale-dependent measures based on squared errors. Let y_i denote the i^{th} observation and $\hat{y}_i$ denote a predicted value of y_i. RMSE is estimated as Equation (8):

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n} \left(y_i - \hat{y}_i\right)^2} \tag{8}$$

3 Results and Discussion

3.1 Logistic Approximation

Table 1 shows the parameter estimates as well as the summary of the logistic model for approximating the population of prisoners in Ghana.

Table 1 Summary of Approximation

Parameter	Estimate	Standard Error
A	0.23	0.03
r	1996.94	0.65
K	14476.93	248.18

AICc=271.5908, RMSE=368.2337, R^2=0.9465

Using the parameter estimates from Table 1, the logistic model for approximating the population of prisoners in Ghana at any time t_i is given by Equation (9).

$$P(t_i) = \frac{14476.93}{1 + \text{Exp}\left[-0.23(t_i - 1996.94)\right]} \tag{9}$$

Fig. 2 shows the observed and the approximated logistic curve for the population of prisoners in Ghana for the next eight years (2025). The model estimates an approximate cumulative population of 14 477 prisoners. This implies that the population of prisoners could reach 14 477 in the near future.

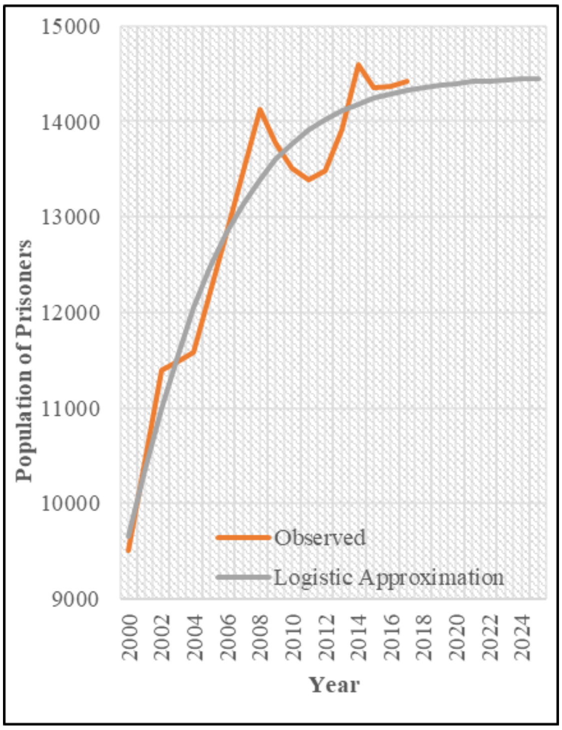

Fig. 2 Observed Annual Population of Prisoners with Approximated Logistic Model

3.2 Gompertz Approximation

Table 2 shows the parameter estimates as well as the summary of the Gompertz model for the population of prisoners in Ghana.

Table 2 Summary of Approximation

Parameter	Estimate	Standard Error
A	0.20	0.03
r	1995.55	0.86
K	14558.12	277.38

AICc=271.5015, RMSE=367.3215, R^2=0.9469

Using the parameter estimates from Table 2, the Gompertz model for approximating the population of prisoners in Ghana at any time t_i is given by Equation (10).

$$P(t_i) = 14558.12 * \text{Exp}\left[-\text{Exp}\left(-0.20(t_i - 1995.55)\right)\right] \tag{10}$$

Fig. 3 shows the observed and the approximated Gompertz curve for the population of prisoners in Ghana for the next eight years (2025). The model estimates approximately 14 558 as the cumulative population of prisoners. This implies that, the population of prisoners in Ghana could increase to about 14 558 in the near future.

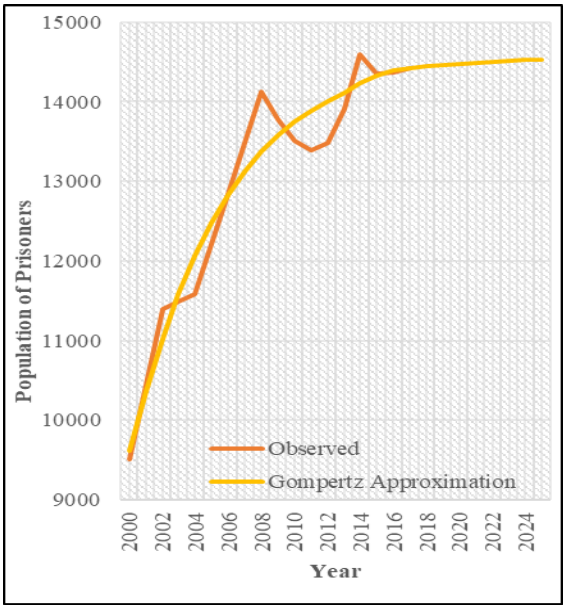

Fig. 3 Observed Annual Population of Prisoners with Approximated Gompertz Model

3.3 Gaussian Approximation

Table 3 shows the parameter estimates as well as the summary of the Gaussian model for approximating the population of prisoners in Ghana.

Table 3 Summary of Approximation

Parameter	Estimate	Standard Error
A	17.16	1.68
r	2014.50	1.18
K	14309.83	158.98

AICc=276.7049, RMSE=424.4423, R^2=0.9290

Using the parameter estimates from Table 3, the Gaussian model for approximating the population of prisoners in Ghana at any time t_i is expressed as in Equation (11).

$$P(t_i) = 14309.83 * \text{Exp}\left[-\left[0.5\left(\frac{(t_i - 2014.50)}{17.16}\right)^2\right]\right] \quad (11)$$

Fig. 4 shows the observed and the approximated Gaussian curve for the population of prisoners in Ghana for the next eight years (2025). The model shows that the cumulative population of prisoners to be approximately 14 310. This implies that, the population of prisoners in Ghana attained its peak at 14 310.

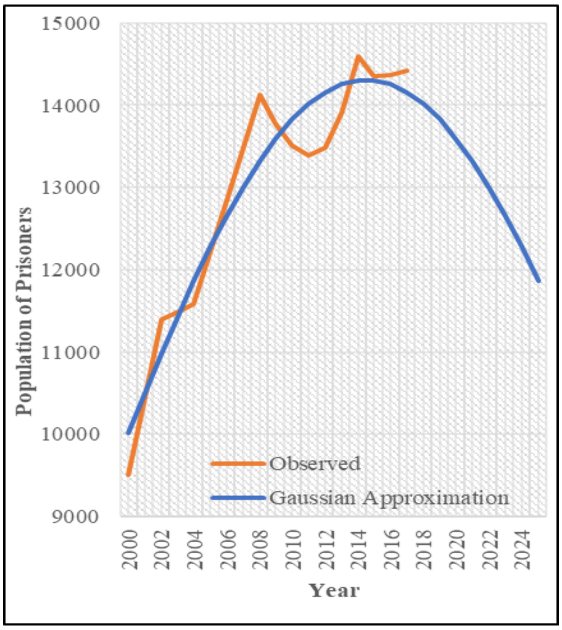

Fig. 4 Observed Annual Population of Prisoners with Approximated Gaussian Model

3.4 Richards Approximation

Table 4 shows the parameter estimates as well as the summary of the Richards model for approximating the population of prisoners in Ghana.

Table 4 Summary of Approximation

Parameter	Estimate	Standard Error
A	0.18	0.15
r	1986.25	149.45
K	14638.66	641.37
L	-44503.69	1378790.90

AICc=275.40, RMSE=379.94, R^2=0.9468

Using the parameter estimates from Table 4, the Richards model for approximating the population of prisoners in Ghana at any time t_i is expressed as in Equation (12).

$$\begin{aligned} P(t_i) &= -44503.69 + (14638.66 + 44503.69) \\ &\quad * \text{Exp}\left[-\text{Exp}\left(-0.18(t_i - 1986.25)\right)\right] \end{aligned} \quad (12)$$

Fig. 5 shows the observed and the approximated Richards curve for the population of prisoners in Ghana for the next eight years (2025). The model estimates an approximate cumulative population of 14 669 prisoners. This implies that, the population of prisoners in Ghana could increase to about 14 669 in the near future.

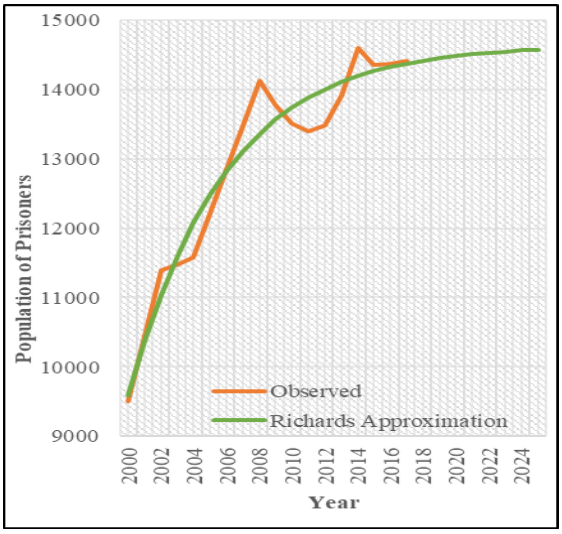

Fig. 5 Observed Annual Population of Prisoners with Approximated Richards Model

Four different population growth models have been approximated based on annual observed data of prisoners in Ghana. Table 5 shows the summary of the approximated models. It can be seen from Table 5 that, all the models approximated yielded a coefficient of determination (R^2) above 0.9200. This implies that all the approximated models are sufficient in term of proportion of variance being explained. Moreover, all the models, with the exception of the Gaussian model suggest that, the population of prisoners in Ghana is bound to increase, since these models have their estimated respective cumulative population greater than the current (2017) observed value of 14 418.

Table 5 Summary of the Approximated Prisoners Population Growth Models

Model	*K*	AICc	RMSE	R^2
Logistic	14476.93	271.59	368.23	0.9465
Gompertz	14558.12	271.50	367.32	0.9469
Gaussian	14309.83	276.70	424.44	0.9290
Richard	14638.66	275.40	379.94	0.9469

Among the approximated models, it can be observed from Table 5 that, the Gompertz model had the least values (in terms of AICc and RMSE). This suggest that, the Gompertz model yielded a better result among the four approximated models. This subsequently led to it achieving the least error value of 367.32 (RMSE). Also, the Gompertz model explains the highest proportion of variance (as indicated by R^2 of 0.9469), suggesting that the Gompertz model approximates closely to the observed trend of population of prisoners in Ghana than any of the other models. Hence, the approximated Gompertz model is selected as the suitable model to represent the dynamics of annual population of prisoners in Ghana.

According to the Gompertz model, the population of prisoners in Ghana could increase to about 14 558 from the current (2017) observed value of 14 418. Table 6 shows the observed and predicted values from the approximated models.

Table 6 Observed and Predicted Values

Year	Observed	Model			
		Logistic	Gompertz	Gaussian	Richard
2000	9507	9656	9620	10014	9590
2001	10449	10357	10364	10501	10373
2002	11390	10993	11017	10975	11039
2003	11486	11558	11583	11432	11604
2004	11581	12052	12069	11867	12082
2005	12214	12477	12483	12277	12486
2006	12847	12838	12833	12658	12828
2007	13488	13140	13127	13006	13116
2008	14128	13391	13373	13319	13359
2009	13778	13599	13579	13593	13563
2010	13507	13768	13750	13826	13735
2011	13396	13907	13892	14015	13880
2012	13487	14019	14009	14159	14001
2013	13908	14110	14107	14255	14103
2014	14599	14183	14229	14304	14189
2015	14355	14242	14329	14304	14262
2016	14368	14289	14389	14255	14322
2017	14418	14327	14419	14159	14373
2018		14357	14444	14015	14416
2019		14381	14464	13826	14452
2020		14401	14481	13594	14482
2021		14416	14495	13320	14507
2022		14428	14506	13007	14528
2023		14438	14516	12658	14546
2024		14446	14523	12277	14561
2025		14452	14530	11868	14573

4 Conclusion

This paper sought to approximate the cumulative population of prisoners in Ghana using four growth models. The models examined were the logistic, Gompertz, Gaussian and the Richards models, which were then used to predict into the future. Although all the models approximated gave a good estimation of the observed data, the Gompertz model was identified to give the best approximation of the observed trend of population of prisoners in Ghana. The Gompertz model suggested that there would be a gradual increase of the population growth of Prisoners in Ghana to a value 14 558 but would not exceed the highest recorded observed value of 145 99 in 2014. Further studies should be conducted to investigate the predominant form of crime cases (drug offenses, fraud, bribery, burglary, homicide, sex offenses *etc*.) that dominates in Ghana prisons.

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